Periodic homogenization of a Lévy-type process with small jumps

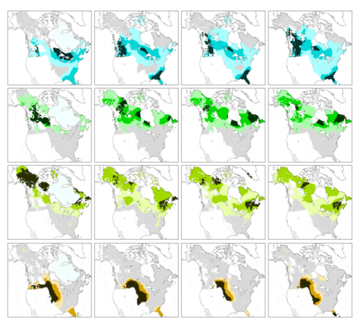
Ivana Valentić, University of Zagreb, Croatia

joint work with N. Sandrić and J. Wang

Modeling

C. Carrillo i P. Fife: Spatial effects in discrete generation population models (2005)

R. Cont i P. Tankov: Financial Modelling with Jump Processes (2004)



J.A. Powell i N.E. Zimmermann: Multiscale analysis of active seed dispersal contributes to resolving Reid's paradox (2004)

Probability approach to homogenization

$$\mathcal{A}_\varepsilon f(x) = \frac{1}{\varepsilon}\left\langle b\left(\frac{x}{\varepsilon}\right), \nabla f(x)\right\rangle + \frac{1}{2} Tr\, c\left(\frac{x}{\varepsilon}\right)\nabla^2 f(x) + \\ + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\left(f(x+\varepsilon y) - f(x) - \varepsilon\langle y, \nabla f(x)\rangle \mathbf{1}_{B_1(0)}(y)\right)\nu\left(\frac{x}{\varepsilon}, dy\right) \\ \longrightarrow \mathcal{A}f(x) \qquad (\varepsilon \searrow 0)$$

infinitesimal generator $\mathcal{A}_\varepsilon \longrightarrow \mathcal{A}$

$\updownarrow \qquad \updownarrow$

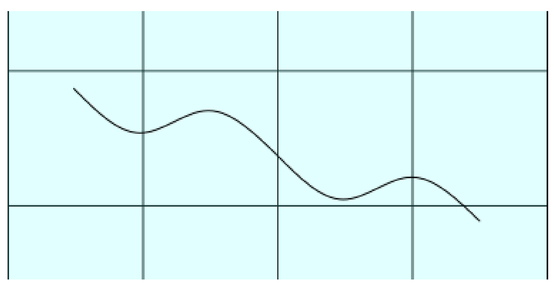
Feller process $X_\varepsilon \longrightarrow X$ (central limit theorem)

Lévy-type process

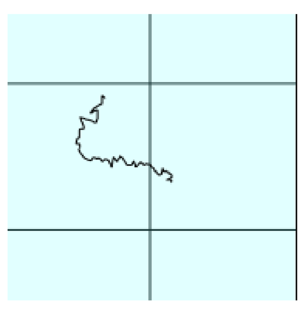
- $\{X_t\}_{t\geq 0}$ **Markov process** on the state space $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$
- $P_t f(x) := \mathbb{E}_x[f(X_t)]$, $t \geq 0$, $x \in \mathbb{R}^d$, $f \in \mathcal{B}_b\left(\mathbb{R}^d\right)$ associated semigroup on $\left(\mathcal{B}_b\left(\mathbb{R}^d\right), \|\cdot\|_\infty\right)$
- $\{X_t\}_{t\geq 0}$ enjoys **Feller property**:
$$P_t\left(C_\infty\left(\mathbb{R}^d\right)\right) \subseteq C_\infty\left(\mathbb{R}^d\right), \text{ for all } t \geq 0,$$
- $\{P_t\}_{t\geq 0}$ is **strongly continuous**:
$$\lim_{t\to 0}\|P_t f - f\|_\infty = 0 \text{ for all } f \in C_\infty\left(\mathbb{R}^d\right)$$
- Infinitesimal generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$, $\mathcal{A}: \mathcal{D}_\mathcal{A} \to \mathcal{B}_b\left(\mathbb{R}^d\right)$
$$\mathcal{A}f := \lim_{t\to 0}\frac{P_t f - f}{t},\; f \in \mathcal{D}_\mathcal{A} := \left\{f \in \mathcal{B}_b\left(\mathbb{R}^d\right) : \lim_{t\to 0}\frac{P_t f - f}{t} \text{ exists w.r.t. } \|\cdot\|_\infty\right\}$$
- $\mathbf{C_c^\infty}\left(\mathbf{R^d}\right) \subseteq \mathcal{D}_\mathcal{A} \Longrightarrow$
$$\mathcal{A}|_{C_c^\infty(\mathbb{R}^d)} f(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}Tr\, c(x)\nabla^2 f(x) + \\ + \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - \langle y, \nabla f(x)\rangle \mathbf{1}_{B_1(0)}(y)\right)\nu(x, dy)$$

Examples

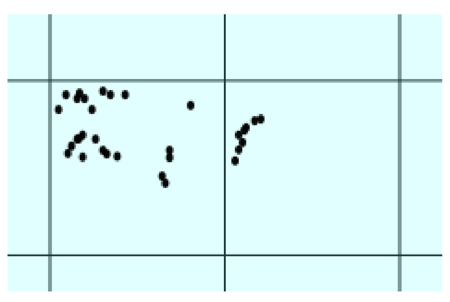
- $c \equiv 0$ and $\nu \equiv 0$
$\Longrightarrow X$ deterministic process



- $\nu \equiv 0$
$\Longrightarrow X$ diffusion process, $\mathcal{A}$ local operator



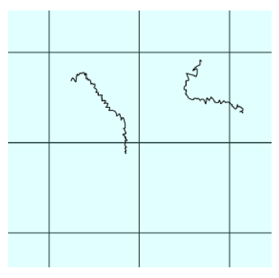
- $b \equiv 0$ and $c \equiv 0$
$\Longrightarrow X$ pure jump process



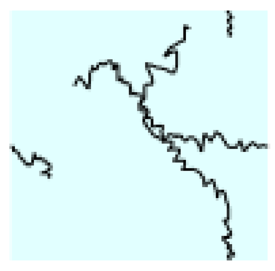
- b, c, ν constant
$\Longrightarrow X$ Lévy process, $\mathcal{A}$ nonlocal operator with constant coefficients
- $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\nu(x, dy) < \infty$
$\Longrightarrow X$ Lévy-type process with "small jumps"

Methods

- projection of the process on the cell of periodicity $\mathbb{R}^d \ni X_t \longrightarrow \Pi(X_t) \in \mathbb{T}$



$\longrightarrow$



- discussion of the stochastic stability property

Proposition

Process $\Pi(X_t)$ admits a unique invariant probability measure π, that is
$$\int_\mathbb{T}\mathbb{P}^x\left(\Pi(X_t) \in B\right)\pi(dx) = \pi(B), \text{ for all } t \geq 0, B \in \mathcal{B}(\mathbb{T})$$

Proposition

Process $\Pi(X_t)$ is geometrically ergodic, that is there exist $\Gamma, \gamma > 0$ such that
$$\sup_{x\in\mathbb{T}}\|\mathbb{P}^x\left(\Pi(X_t) \in dy\right) - \pi(dy)\|_{TV} \leq \Gamma e^{-\gamma t}, \text{ for all } t \geq 0,$$
where $\|\cdot\|_{TV}$ is a total variation norm.

- central limit theorem

Theorem

$$\left\{\varepsilon X_{\varepsilon^{-2}t} - \varepsilon^{-1}\bar{b}^* t\right\}_{t\geq 0} \overset{\varepsilon\to 0}{\Longrightarrow} \{W_t\}_{t\geq 0},$$
where $b^*(x) = b(x) - \int_{B_1^c(0)} y\nu(x, dy)$ and $\bar{b}^* := \int_\mathbb{T} b^*(x)\pi(dx)$, and W_t Brownian motion determined by covariance matrix Σ given in terms of coefficients b, c and ν.

Additional assumptions

This method can be applied for process $\{X_t\}_{t\geq 0}$ if

- $(b(x), c(x), \nu(x, dy))$ **periodic** the cell of periodicity $\mathbb{T}$,
- $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\nu(x, dy) < \infty$,
- $\{X_t\}_{t\geq 0}$ satisfies **strong Feller property**: $P_t\left(\mathcal{B}_b\left(\mathbb{R}^d\right)\right) \subseteq C_b\left(\mathbb{R}^d\right)$ for all $t > 0$,
- $\{X_t\}_{t\geq 0}$ is **open set irreducible**: $\mathbb{P}_x(X_t \in O) > 0$ for all $t > 0$, all $x \in \mathbb{R}^d$ and all nonempty open $O \subseteq \mathbb{R}^d$,
- $x \mapsto b^*(x)$ is in $C_b^\psi\left(\mathbb{R}^d\right)$ for some Hölder exponent ψ,
- for some $t_0 > 0$, all $t \in \langle 0, t_o]$ and all $\mathbb{T}$-periodic $f \in C_b\left(\mathbb{R}^d\right)$ there exists $C(t)$ s.t. $\|P_t f\|_\psi \leq C(t)\|f\|_\infty$ i $\int_0^{t_0} C(t)dt < \infty$,
- for some $\lambda > 0$ and all $\mathbb{T}$-periodic $f \in C_b^\psi\left(\mathbb{R}^d\right)$ s.t. $\int_\mathbb{T} f(x)\pi(dx) = 0$ Poisson equation
$$\lambda u - \mathcal{A}u = f$$
admits a unique $\mathbb{T}$-periodic solution $u_{\lambda,f} \in C_b^{\rho\psi}\left(\mathbb{R}^d\right)$ for some Hölder exponent ρ

References

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